Stability, interpretability and nonprojectibility

Jayde Massmann

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Outline

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Reflection principles

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Reflection principles and inaccessible cardinals turn out to be a major driving theme in modern set theory.

- The universe of all sets is stratified via the von Neumann hierarchy $\langle V_\alpha : \alpha \in \mathrm{Ord}\rangle$.
- Lévy-Montague reflection: even without having to assume the existence of inaccessible cardinals etc, Ord has powerful reflection properties.
- For any first-order formula, any set is contained in some object which "reflects" that formula, i.e. satisfies it iff the whole universe does.

Formal reflection

If κ is an inaccessible cardinal, then V_κ satisfies all the axioms of ordinary set theory, ZFC, and so this reflection principle holds inside V_κ. We can actually do better, by permitting special predicate symbols (i.e. extending the language of set theory by a "black box" oracle).

Definition

Let $n, m < \omega$. Then a cardinal κ is called Π^n_m-indescribable if, whenever φ is a $\Pi_m(\dot{A})$ sentence and $A \subseteq V_\kappa$ is so that $\langle V_{\kappa+n}, \in, A\rangle \models \varphi$, there is $\alpha < \kappa$ so that $\langle V_{\alpha+n}, \in, A \cap V_\alpha\rangle \models \varphi$.

Notes

n is meant to be the "order", as e.g. a first-order sentence over $V_{\kappa+1}$ becomes a second-order sentence over V_κ. So the following are basically equivalent: κ is Π^n_0-indescribable, and the n-th order reflection principle with a predicate holds in V_κ. In particular:

Lemma

κ is inaccessible iff it is Π^1_0-indescribable.

However, for all $n < \omega$, the least Π^1_n-indescribable cardinal is not Π^1_{n+1}-indescribable.

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Admissible sets

- At the countable level, analogues to regular or indescribable cardinals can be obtained by stratifying our universe via the constructible hierarchy instead.
- Say M is admissible if all axioms of Kripke-Platek set theory hold within M.
- For example, $V_{\omega+\omega}$ satisfies all axioms of Kripke-Platek set theory except Δ_0-collection.

Admissible ordinals

- In general, if V_κ is admissible, then $\beth_\kappa = \kappa$.
- Meanwhile, by using a "predicative" construction, and in particular not adding all possible subsets at each next stage, we'll be immune to this counterexample.
- The smallest admissible ordinal is equal to ω_1^{CK}. This is the supremum of order-types of computable well-orders of ω.

Generalized recursion theory

Say $A \subseteq L_\alpha$ is α-recursive (resp. α-recursively enumerable) if it is $\Delta_1(L_\alpha)$ (resp. $\Sigma_1(L_\alpha)$). The reason for the naming "α-recursive" is that a subset of ω is recursive (i.e. computable) iff it is Δ^0_1 in the arithmetical hierarchy, and recursively enumerable iff it is Σ^0_1.

Recursive analogues

Admissible ordinals are traditionally considered as a "recursive analogue" of regular cardinals because α is admissible iff $\alpha > \omega$, α is a limit ordinal and either of the following hold:

- For all $\delta < \alpha$, there is no α-recursively enumerable map $\delta \to \alpha$ with cofinal range.
- For all $\delta < \alpha$, there is no α-recursive surjection $\delta \to \alpha$.

This suggests attempting to generalize other large cardinal axioms to the countable level. Π^1_n-indescribability in particular can be copied almost verbatim, although one has to replace the predicate A with a parameter $b \in L_\alpha$.

Reflecting ordinals

Definition

An ordinal α is Π_n-reflecting if $\alpha > 0$ and, whenever φ is a Π_n-formula and $b \in L_\alpha$ is so that $L_\alpha \models \varphi(b)$, there is $\beta < \alpha$ so that $b \in L_\beta$ and $L_\beta \models \varphi(b)$.

It's immediate to see that any Π_n-reflecting ordinal is a limit ordinal. Due to downwards absoluteness, Π_1-reflection is actually equivalent to being a limit.

$n > 1$

- Meanwhile, Π_2-reflection is equivalent to admissibility.
- For $n > 2$, any Π_n-reflecting ordinal is a limit of Π_2-reflecting ordinals.
- In general, for $n > 0$, the recursive analogue of Π^1_n-indescribability may be considered to be Π_{n+2}-reflection.

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Transfinite extensions

- My work primarily focuses on transfinite extensions of these, i.e. notions beyond Π^n_m-indescribability or Π_n-reflection for $n, m < \omega$.
- Unfortunately, directly generalizing indescribability or reflection (e.g. simply considering Π^α_m-indescribability for arbitrary ordinals α, without changing the definition) doesn't work.
- Instead, we need to slightly modify the definitions. The three notions given in the following definition are, among nonprojectibility and subtlety, going to be the key ones.

Shrewdness, transfinite reflection and stability

Definition

Let $\eta > 0$. A cardinal κ is called η-shrewd iff, for all $P \subseteq V_\kappa$ and every formula $\varphi(x)$ (possibly using the new predicate $\dot{A}$), if $\langle V_{\kappa+\eta}, \in, P\rangle \models \varphi(\kappa)$, then there exist $0 < \kappa_0, \eta_0 < \kappa$ so that $\langle V_{\kappa_0+\eta_0}, \in, P \cap V_{\kappa_0}\rangle \models \varphi(\kappa_0)$.

An ordinal α is called ξ-Π_n-reflecting iff, for all $b \in L_\alpha$ and every Π_n-formula $\varphi(x)$, if $L_{\alpha+\xi} \models \varphi(b)$, then there exist $\alpha_0, \xi_0 < \alpha$ so that $b \in L_{\alpha_0}$ and $L_{\alpha_0+\xi_0} \models \varphi(b)$.

Let $\xi > 0$. An ordinal α is called ξ-stable iff, for all $b \in L_\alpha$ and every Σ_1-formula $\varphi(x)$, $L_\alpha \models \varphi(b)$ iff $L_{\alpha+\xi} \models \varphi(b)$.

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$\mathcal{A}$-stability

Naturally, one may consider adding back an oracle.

Definition

Let $\mathcal{A}$ be an arbitrary class and let $\xi > 0$. An ordinal α is called $\mathcal{A}$-ξ-stable iff, for all $b \in L_\alpha$ and every $\Sigma_1(\dot{A})$-formula $\varphi(x)$, $\langle L_\alpha, \in, \mathcal{A} \cap L_\alpha \rangle \models \varphi(b)$ iff $\langle L_{\alpha+\xi}, \in, \mathcal{A} \cap L_{\alpha+\xi} \rangle \models \varphi(b)$.

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This notion is trivial when $\mathcal{A} \subseteq L_\alpha$, and obviously most interesting when $\mathcal{A} \subseteq L_{\alpha+\xi}$.

An example

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This notion can attain exorbitant strength – for example, let $\xi > \alpha$, $L_\xi \models$ ZFC and $\mathcal{A}$ be the set of $\beta < \xi$ which are ξ-stable. Then α is $\mathcal{A}$-ξ-stable iff the ordinary definition of ξ-stability holds for α, but where φ is allowed to Σ_2. That is, $\Sigma_1(\dot{A})$-substructurehood in this case paves the way for full Σ_2-substructurehood. Σ_2-stability is actually possibly more relevant to our work than Σ_1-stability.

What are subtle cardinals?

Subtle cardinals were introduced by Ronald Jensen in his analysis of the fine structure of L.

Definition

A cardinal κ is subtle if the following holds. Let $C \subseteq \kappa$ be an arbitrary club (closed and unbounded) set. And assume $\vec{S} : C \to \mathcal{P}(\kappa)$ is regressive, in that $\vec{S}(\alpha) \subseteq \alpha$ for all $\alpha \in C$. Then there are $\beta, \delta \in C$ so that $\beta < \delta$ and $\vec{S}(\delta) \cap \beta = \vec{S}(\beta)$.

If κ is subtle, then $\Diamond_\kappa$ holds.

Why did we mention them?

- It's known that subtle cardinals are "small" large cardinals.
- They're below Ramsey, measurable, etc. cardinals (and other ultrafilter- or elementary embedding-based large cardinals) in terms of consistency strength.
- But they're above Π^n_m-indescribable cardinals for all $n, m < \omega$.
- So it's natural to try to also calibrate a recursive analogue of them.
- It turns out: there's a characterisation of subtlety via shrewdness!

Extra oracles for shrewdness...

As one may expect, this characterisation involves adding an extra oracle for shrewdness.

Definition

Let $\mathcal{A}$ be an arbitrary class and let $\eta > 0$. A cardinal κ is called $\mathcal{A}$-η-shrewd iff, for all $P \subseteq V_\kappa$ and every formula $\varphi(x)$ (possibly using new predicates $\dot{P}$ and $\dot{A}$), if $\langle V_{\kappa+\eta}, \in, P, \mathcal{A} \cap V_{\kappa+\eta} \rangle \models \varphi(\kappa)$, then there exist $0 < \kappa_0, \eta_0 < \kappa$ so that $\langle V_{\kappa_0+\eta_0}, \in, P \cap V_{\kappa_0}, \mathcal{A} \cap V_{\kappa_0+\eta_0} \rangle \models \varphi(\kappa_0)$.

Extra oracles for shrewdness...

We show later that stability with oracles may be considered as a recursive analogue of this version of shrewdness with oracles. For now, let us first give the promised characterisation of subtlety.

Theorem

A strongly inaccessible cardinal π is subtle iff the following holds: for any $\mathcal{A} \subseteq V_\pi$, there are stationarily many $\kappa < \pi$ so that κ is $\mathcal{A}$-η-shrewd for all $\eta < \pi$.

Recursive subtlety

This motivates the following definition:

Definition

ρ is recursively subtle iff, for any ρ-recursively enumerable $\mathcal{A} \subseteq L_\rho$, ρ is Π_2-reflecting onto the set of $\kappa < \rho$ which are $\mathcal{A}$-ρ-stable.

Our main theorem is that recursively subtle ordinals are precisely Σ_2-nonprojectible ordinals.

Nonprojectibility

Theorem

Suppose $\beta < \omega_1^L$. Then the following are equivalent:

- For all $\tau < \beta$, there is $\alpha < \beta$ so that $\tau < \alpha$ and α is β-Σ_2-stable.
- The Σ_2-projectum ρ_2^β is equal to β.
- If $\mathcal{A} \subseteq \beta$ is $\Sigma_2(L_\beta)$ and $\sup(\mathcal{A}) < \beta$, then in fact $\mathcal{A} \in L_\beta$.

Nonprojectibility

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Suppose $\beta < \omega_1^L$. Then the following are equivalent:

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- The Σ_2-projectum ρ_2^β is equal to β.
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Any ordinal satisfying any of these equivalent conditions is called Σ_2-nonprojectible (due to condition 2), although only formulation 1 will be relevant for our purposes.

The crux of the proof

Our goal is so-called oracle elimination: the title of this section. Basically, we want to show that, in $\mathcal{A}$-ξ-stability, the additional oracle $\mathcal{A}$ is irrelevant given higher degrees of correctness: any ξ-Σ_2-stable ordinal is $\mathcal{A}$-ξ-stable.

The crux of the proof

Our goal is so-called oracle elimination: the title of this section. Basically, we want to show that, in $\mathcal{A}$-ξ-stability, the additional oracle $\mathcal{A}$ is irrelevant given higher degrees of correctness: any ξ-Σ_2-stable ordinal is $\mathcal{A}$-ξ-stable. We don't want to do this for all $\mathcal{A}$, only the ξ-recursively enumerable $\mathcal{A}$ (and a bit beyond, including the ξ-co-recursively enumerable sets). Without stringent conditions, full oracle elimination is certainly not possible.

Interpretability

Definition

Let κ, ρ be ordinals with $\kappa < \rho$. Then $\mathcal{A} \subseteq L_\kappa$ is called (κ, ρ)-interpretable iff, for any Σ_2 formula φ and parameters $\vec{b} \in L_\kappa$ with $\mathcal{A} = \{x \in L_\kappa : L_\kappa \models \varphi(x, \vec{b})\}$, we have $\mathcal{A} = \mathcal{A}^{L_\rho} \cap L_\kappa$, where $\mathcal{A}^{L_\rho} = \{x \in L_\rho : L_\rho \models \varphi(x, \vec{b})\}$.

Global interpretability from stability

In other words, $\mathcal{A}$ is (κ, ρ)-interpretable iff it any Σ_2-definition gives a way of extending $\mathcal{A}$ to a "fuller" subset of L_ρ without adding new elements of L_κ.

Lemma

Let κ, ξ be ordinals so that $\xi > 0$. Then the following are equivalent:

- κ is ξ-Σ_2-stable.
- Every subset of L_κ is $(\kappa, \kappa + \xi)$-interpretable.

Oracle elimination

Using interpretability, you can derive $\Delta_2(\dot{A})$-preservation from Σ_2-preservation, i.e. oracle elimination:

Theorem

Let κ, ξ be ordinals so that $\xi > 0$. Assume κ is ξ-Σ_2-stable and $\mathcal{A} \subseteq L_{\kappa+\xi}$ is Δ_2-definable in $L_{\kappa+\xi}$ with parameters from L_κ. Then κ is $\mathcal{A}$-ξ-stable.

How to proceed

Anyways, now we have the following: assume ρ is Σ_2-nonprojectible. Recall characterisation 1 – for all $\tau < \rho$, there is $\alpha < \rho$ so that $\tau < \alpha$ and α is ρ-Σ_2-stable. A relatively easy (yet surprising) reflection argument shows that, not only is β a limit of β-Σ_2-stable ordinals, but actually β is Π_2-reflecting on them. Therefore, for the conclusion that ρ is recursively subtle, it suffices to prove that a tail of ρ-stable ordinals are also $\mathcal{A}$-ρ-stable. And to do this, simply pick a ρ-stable ordinal κ so that $\mathcal{A}$ is definable in L_ρ with parameters from L_κ.

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Recursive measurability?

α is Σ_2-extendible iff there is $\beta > \alpha$ so that α is β-Σ_2-stable.

The smallest Σ_2-extendible ordinal has a characterisation in terms of infinite-time computability, and arithmetical quasi-inductiveness. It was previously also believed that Σ_2-extendibility may serve as a recursive analogue of measurability.

Evidence?

Say α is recursively measurable iff there is a nonprincipal ultrafilter on the Boolean algebra of α-recursive subsets of α, which is closed under intersections of $< \alpha$ many sets, as long as this intersection can be coded in an α-recursive way. Then α is recursively measurable iff it is Σ_2-extendible.

Evidence?

Say α is recursively measurable iff there is a nonprincipal ultrafilter on the Boolean algebra of α-recursive subsets of α, which is closed under intersections of $< \alpha$ many sets, as long as this intersection can be coded in an α-recursive way. Then α is recursively measurable iff it is Σ_2-extendible. This assignment is however not "consistent" with our findings: any measurable cardinal is weakly compact and subtle, but:

Proposition

If α is Σ_2-extendible, it is Π_3-reflecting and nonprojectible, yet the least Σ_2-extendible ordinal is not Σ_2-nonprojectible.

Remark

Maybe this dissimilarity occurs because there can be no true recursive analogue of measurability. After all, a main application of recursive analogues is in ordinal analysis, which can be carried out absolutely and hence measurable cardinals will never be needed for it.

Thanks!

Thanks for listening!