$\mathcal{P}(\kappa)/\mathrm{NS}_\kappa$ and Stationary Reflection

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Outline

Table of Contents

Large cardinals

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- Inaccessible, Mahlo, weakly compact, subtle, ineffable (comfortably compatible with $V = L$)
- Measurable, strong, Woodin, superstrong (suddenly, L must be far from V)
- Supercompact, huge, Reinhardt (we have no good theory of the universe's "fine structure" here)

Where do we want to focus?

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However, today I'm really interested in what goes down at the lower levels: around a weakly compact cardinal, at most an ineffable.

The Nonstationary Algebra

Let $\mathrm{NS}_\kappa \subseteq \mathcal{P}(\kappa)$ be the ideal of nonstationary subsets of κ - i.e. those disjoint from some closed unbounded set. $\mathcal{P}(\kappa)/\mathrm{NS}_\kappa$ consists of equivalence classes under $X \sim Y$ iff $X \triangle Y \in \mathrm{NS}_\kappa$. $[X] \leq [Y]$ iff $X \setminus Y \in \mathrm{NS}_\kappa$.

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- The bottom element, $\mathbf{0} = [\emptyset] = \mathrm{NS}_\kappa$ is the nonstationary ideal.
- The top element, $\mathbf{1} = [\kappa] = \mathrm{NS}_\kappa^*$ is the club filter.
- Equivalence classes inbetween consist solely of stationary, costationary sets (not disjoint from any club but don't contain one either).

Antichains

Recall a (strong) antichain in a BA is a set of pairwise incompatible elements, where x, y are *incompatible* if $x \wedge y = \mathbf{0}$, ignoring $x, y = \mathbf{0}$.

Theorem

(Solovay) For any stationary $S \subseteq \kappa$, there is a family $\{X_\alpha : \alpha < \kappa\} \subseteq \mathcal{P}(S)$ so that X_α is stationary and $X_\alpha \cap X_\beta = \emptyset$ for $\alpha \neq \emptyset$; in particular $\mathcal{P}(\kappa)/\,\mathrm{NS}_\kappa$ has size-κ antichains below every nonzero element.

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Question

Does $\mathcal{P}(\kappa)/\operatorname{NS}_\kappa$ have size-κ^+ antichains?

Yes, usually

Theorem

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$V = L$ implies yes for $\kappa = \omega_1$ too, since it implies a strong form of CH, $\Diamond$. Whether CH is enough is open. Notice that $\mathcal{P}(\kappa)/\operatorname{NS}_\kappa$ has a size-λ antichain if it has a length-λ descending chain.

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The items of a descending chain get "thinner" as time goes on, so there being a long one while everything remains stationary is indicative of κ being "large". Indeed, the first proof that NS_κ must fail to be "κ^+-saturated" (Baumgartner-Taylor-Wagon) is when κ is "greatly Mahlo", which follows from weak compactness.

Table of Contents

Stationary reflection

However, $[X_\beta] \leq [X_\alpha]$, $X_\beta \not\sim X_\alpha$ isn't "strong enough" for purposes of actually defining large cardinals, which is what I'm interested in here.

Definition

An $X \notin \mathrm{NS}_\kappa$ *reflects* if $X \cap \alpha \notin \mathrm{NS}_\alpha$ for some $\alpha < \kappa$. It reflects stationarily often if $\mathrm{Tr}(X) = \{\alpha < \kappa : X \cap \alpha \notin \mathrm{NS}_\alpha\} \notin \mathrm{NS}_\kappa$.

Clearly, $X \subseteq \kappa^+$ cannot reflect if it consists entirely of cofinality-κ ordinals ($X \subseteq S^{\kappa^+}_\kappa$). Generally, $\mathrm{Tr}(X)$ is a lot thinner than X.

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Theorem

(Folklore) The statement "every stationary $S \subseteq S^{\omega_2}_\omega$ reflects" is equiconsistent with a Mahlo cardinal.

Stationary reflection at limit cardinals

Another interesting result *re* large cardinals:

Theorem

(Jensen, Kunen) If $V = L$, κ is weakly compact iff every stationary subset of κ reflects. The converse can fail if $V \neq L$ (even if $V = L[A]$ for $A \subseteq \kappa$).

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A priori, there is no reason for κ to be more than ω-Mahlo when every stationary subset of κ reflects, where κ is 0-Mahlo (Mahlo) if the set of regular cardinals below is stationary, and κ is $\alpha + 1$-Mahlo if the set of α-Mahlo cardinals below is stationary. Take conjunctions at limit α.

The Mahlo order

Indeed, ω-Mahloness is optimal, by forcing to "kill" $\{\alpha < \kappa : \alpha \text{ is } \omega\text{-Mahlo}\}$ while preserving "every stationary subset reflects" (this is folklore, but this is not how Kunen's original proof went). Weakly compact cardinals are κ-Mahlo and more.

Now put $X < Y$ iff X, Y are stationary and $[Y] \leq [\mathrm{Tr}(X)]$. Notice the order is reversed, so any club is now the least element, not the greatest.

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Now put $X < Y$ iff X, Y are stationary and $[Y] \leq [\mathrm{Tr}(X)]$. Notice the order is reversed, so any club is now the least element, not the greatest. Surprisingly:

Theorem

(Jech) $<$ is well-founded.

Example

$$S^\kappa_\omega < S^\kappa_{\omega_1} < S^\kappa_{\omega_2} < \cdots < \{\alpha < \kappa : \mathrm{cof}(\alpha) = \alpha\}$$
$$< \{\alpha < \kappa : \alpha \text{ is Mahlo}\} < \cdots < \{\alpha < \kappa : \alpha \text{ is } \alpha\text{-Mahlo}\} < \cdots$$

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- $o(\kappa) \geq \kappa$ iff κ is (weakly) inaccessible.
- $o(\kappa) > \kappa$ iff κ is Mahlo.

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- $o(\aleph_{\alpha+1}) = \alpha + 1$.
- $o(\kappa) \geq \kappa$ iff κ is (weakly) inaccessible.
- $o(\kappa) > \kappa$ iff κ is Mahlo.
- At $\kappa \cdot \omega$, we "catch up" with if we only allow regular reflection points, which would make S^κ_μ have rank 0 for all fixed $\mu < \kappa$.

Table of Contents

Weakly compact cardinals

Clearly, $o(\kappa) < (2^\kappa)^+$. Recall that if κ is weakly compact then κ is κ-Mahlo, so $o(\kappa) \geq \kappa \cdot 2$. In fact, $o(\kappa) > \kappa^+$. So, if $2^\kappa = \kappa^+$, we're in the awkward position:

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Meanwhile, if $2^\kappa > \kappa^{++}$, can we have e.g. $o(\kappa) = \kappa^{++}$ or $o(\kappa) > \kappa^{++}$? Test question:

Question

If κ is weakly compact, must there be a forcing extension with $o(\kappa) \geq \kappa^{++}$, all cardinals preserved, and the continuum function untouched except at κ?

The Club Domination Order

For $f, g : \kappa \to \kappa$, $f \leq^* g$ iff $\{\alpha < \kappa : f(\alpha) \leq g(\alpha)\}$ contains a club. If $\mathrm{cof}(\kappa) > \omega$ then this is a well-founded partial order, by countable completeness of the club filter. Let ᚠ(κ) denote its height. Clearly, $\mathrm{cof}($ᚠ$(\kappa)) > \kappa$ and ᚠ$(\kappa) < (2^\kappa)^+$.

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- (Donder-Levinski) *Weak Chang's Conjecture*, aka *Club Bounding*: $\digamma(\omega_1) = \omega_2$. This is slightly below an ω_1-Erdős – between an ineffable and a measurable – in consistency strength.
- (Donder-Koepke) For $\kappa > \omega_1$, $\digamma(\kappa) = \kappa^+$ implies $0^\dagger$ (zero-dagger, between a measurable and two measurables in consistency strength) exists. No upper bound is known, to my knowledge.

Better Bounds on a Weakly Compact Cardinal's Mahlo Rank

The observation of a weakly compact cardinal's Mahlo rank and these facts about ᚠ leads to a natural conjecture.

Conjecture

If κ is weakly compact, then $o(\kappa) \geq$ ᚠ(κ).

The natural way to go about this is a transfinite recursion, translating a $\leq^*$-increasing chain of functions into a $<$-increasing chain of stationary sets. But I haven't gone through the details yet.

Forcing

Observe that it's quite easy to change $\text{ᚠ}(\kappa)$ by forcing. Specifically, κ-Hechler forcing can be iterated with $<\kappa$-support for γ stages to make $\text{ᚠ}(\kappa)^{V[G]} \geq \text{ᚠ}(\kappa)^V + \gamma$ (in fact $\text{ᚠ}(\kappa)^{V[G]} \geq \text{ᚠ}(\kappa)^V \cdot (1+\gamma)$ and much more) while preserving all cardinals and cofinalities (note that the κ^+-cc alone isn't necessarily preserved in infinite-support iterations, we need a form of centeredness).

Killing Weak Compactness

When trying to build a model with large $o(\kappa)$ or $\text{ᚠ}(\kappa)$, it seems possible that κ loses its weak compactness, making it uninteresting in the extension. Some preparations exist to remedy this:

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- (Hamkins) In fact, it is possible to arrange for a cardinal's weak compactness to be killed by any $< \kappa$-closed forcing, the antithesis to Laver's result.

An Altered Test Question

I'm not sure what preparation to use when trying to increase $o(\kappa)$, ᚠ(κ) by forcing.

The test question becomes interesting if we stipulate that you can't increase ᚠ(κ) while trying to increase $o(\kappa)$, e.g. starting with $\kappa^+ <$ ᚠ$(\kappa) \leq o(\kappa) < \kappa^{++}$ and then obtaining $\kappa^+ <$ ᚠ$(\kappa) < \kappa^{++} \leq o(\kappa)$.

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Thank you!

However, these increase $ᚠ(\kappa)$ and may not be sufficiently iterable with $< \kappa$ support. Possibly one could generalize some other poset for adding a new real – such as Miller forcing – or come up with an entirely new idea. Maybe I'm missing something obvious.

Thanks for listening.